Mechanics IV

FIZIKA SJPO Training

4 April 2026

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1 Notes

1.1 Rotational Mechanics

Now, we shall study how objects rotate. So far, our discussion has been restricted to **point masses,** which have all the mass concentrated at one single point. Hence, we didn't need to consider rotation. In real life, nothing is a point mass, and everything has some finite size. Hence, we need to consider rotation for these objects, which we call **rigid bodies.** We shall study both the kinematics and the dynamics of rigid bodies.

1.1.1 Rotational Kinematics

In regular (translational) kinematics, we dealt with the SUVAT equations. Rotational kinematics is actually very similar - we will also be dealing with a set of equations.

The first important thing is how we can make an analogy between translational and rotational quantities, so that we can write corresponding "SUVAT equations" for rotational quantities. The analogy is as such:

Linear	Angular
Translation	Rotation
Position ($\vec{s}$)	Angular Position ($\vec{\theta}$)
Displacement ($\Delta\vec{s}$)	Angular Displacement ($\Delta\vec{\theta}$)
Velocity ($\vec{v}$)	Angular Velocity ($\vec{\omega}$)
Acceleration ($\vec{a}$)	Angular Acceleration ($\vec{\alpha}$)
Force ($\vec{F}$)	Torque ($\vec{\tau}$)
Mass (m)	Moment of Inertia (I) Rotational Inertia
$\vec{F}_{net} = m\vec{a}$	$\vec{\tau}_{net} = I\vec{\alpha}$
Translational KE ($\frac{1}{2}m\vec{v}^2$); $\text{WD}_\text{F} = \vec{F}\Delta\vec{s}$	Rotational KE ($\frac{1}{2}I\vec{\omega}^2$) $\text{WD}_\text{T} = \tau\Delta\vec{\theta}$
Linear Momentum ($\vec{p} = m\vec{v}$), $\vec{J} = \vec{F}\Delta t$	Angular Momentum ($\vec{l} = \text{I}\vec{\omega}$), $\overrightarrow{\text{Angular Impluse}} = \tau\Delta t$
Linear Condition for Equilibrium $\vec{F}_{net} = 0$	Angular Condition for Equilibrium $\vec{\tau}_{net} = 0$

Some quantities here might be foreign to you, but don't worry yet!

In any case, we can see that the "SUVAT equations" are:

$$\omega_\text{f} = \omega_\text{i} + \alpha t \tag{1}$$

$$\theta_\text{f} = \theta_\text{i} + \omega_\text{i} t + \frac{1}{2}\alpha t^2 \tag{2}$$

$$\omega_\text{f}^2 = \omega_\text{i}^2 + 2\alpha\theta \tag{3}$$

$$\theta = \frac{1}{2}\left(\omega_\text{i} + \omega_f\right) t \tag{4}$$

Example 1.1. You have just finished watching a movie on DVD and the disc is slowing to a stop. The angular velocity of the disc at $t = 0\,\text{s}$ is $27.5\,\text{rad/s}$ and its angular acceleration is a constant $-10.0\,\text{rad/s}^2$. A line PQ on the surface of the disc lies along the positive x-axis at $t = 0\,\text{s}$. (a) What is the disc's angular velocity at $t = 0.3\,\text{s}$? (b) What angle does the line PQ make with the positive x-axis at this time?

Solution. (a) Applying Equation (1), we have:

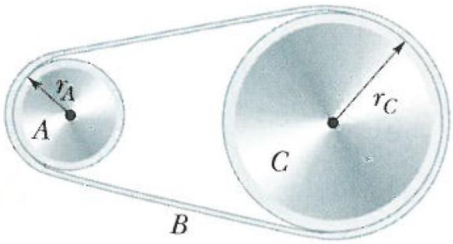
$$\omega_\mathrm{f} = \omega_\mathrm{i} + \alpha t = 27.5 + (-10.0)\,(0.3) = 24.5\,\mathrm{rad/s}$$

(b) Applying Equation (2), we have:

$$\theta = \omega_\mathrm{i} t + \frac{1}{2}\alpha t^2 = (27.5)\,(0.3) + \frac{1}{2}\,(-10.0)\,(0.3)^2 = 7.8\,\mathrm{rad} = 446.9^\circ$$

However, since angles are periodic every 360°, the true angle is actually 86.9°!

Example 1.2. In the conveyor belt system below, $r_\mathrm{A} = 10\,\mathrm{cm}$ and $r_\mathrm{C} = 25\,\mathrm{cm}$. Wheel A increases its angular speed from rest at a uniform rate of $1.6\,\mathrm{rad/s^2}$. How much time is needed for wheel C to reach a rotational speed of 100 rev/min?



Solution. First, we need to identify what is common between wheels A and C. Observe that the tangential speed must be the same, so that the belt B can move as a whole piece. Hence, we have:

$$v_\mathrm{A} = v_\mathrm{C} \quad \Longrightarrow \quad r_\mathrm{A}\omega_\mathrm{A} = r_\mathrm{C}\omega_\mathrm{C}$$

When wheel C reaches a rotational speed of $100\,\mathrm{rev/min} = 100 \times \frac{2\pi}{60}\,\mathrm{rad/s} = \frac{10}{3}\pi\,\mathrm{rad/s}$, this means that A has reached a rotational speed of

$$\omega_\mathrm{A} = \frac{r_\mathrm{C}\omega_\mathrm{C}}{r_\mathrm{A}} = \frac{(25)\left(\frac{10}{3}\pi\right)}{10} = 26.2\,\mathrm{rad/s}$$

Finally, applying Equation (1), we have:

$$t = \frac{\omega_A}{\alpha_\mathrm{A}} = \frac{26.2}{1.6} = 16.4\,\mathrm{s}$$

1.1.2 Moment Of Inertia

The **moment of inertia** (or MOI), I, is the rotational analogue of the mass m. It is a measure of an object's resistance to changes in its rotation rate.

For a point mass m at a distance r away from a rotational axis, its associated MOI is:

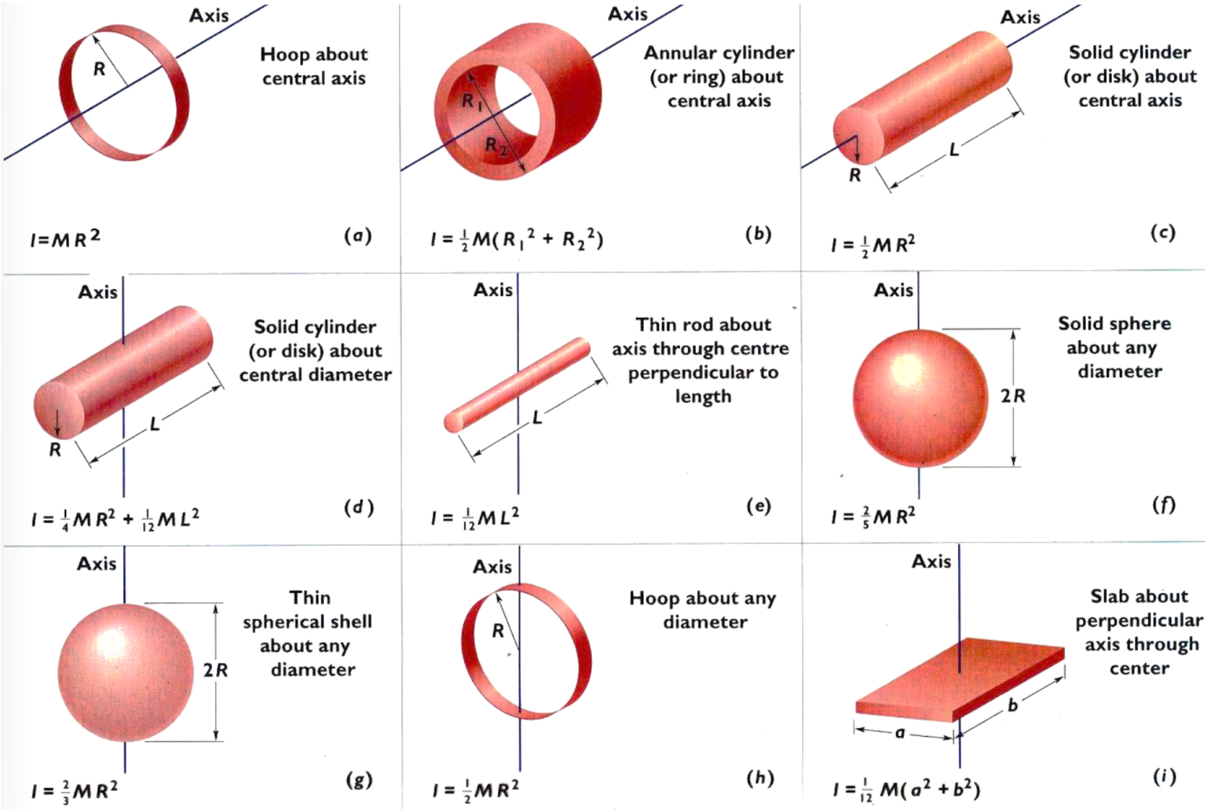
$$I_\text{point mass} = mr^2 \tag{5}$$

As such, if we have a collection of **discrete** point masses m_i at distances r_i from a rotational axis, the total MOI is:

$$I_\text{point masses} = \sum_i m_i {r_i}^2 \tag{6}$$

Clearly, the MOI for the same total mass is dependent on **how the mass is distributed.** If more mass is further away from the axis, then the MOI is generally larger, which means it is harder to rotate the object.

In general, calculating the MOI for a **continuous mass distribution** (where masses are not discrete point masses) is a difficult task that requires integration (outside the scope of SJPO). However, you will most frequently encounter the most common MOIs as stated below:



There are a few noteworthy differences between some of these shapes:

1. A **thin rod** means the radius of the rod is negligible compared to its length, while this approximation is not true for a **solid cylinder/disk.**
2. A **solid sphere** means the mass is uniformly distributed over the entire volume, while a **thin spherical shell** means the mass is uniformly distributed over the entire surface area.

Remark. You are generally not expected to memorise these (a well-set question appropriate for the SJPO will always provide any MOIs that you need), but it is still a good idea to try to remember them (which should come naturally with more practice), just in case.

1.1.3 Parallel Axis Theorem

When it comes to evaluating MOI, the **parallel axis theorem** is a powerful tool, if we already know the MOI of an object about an **axis passing through its CM.**

Let I be the MOI of an object (of mass m) about an axis located a distance d away from a parallel axis passing through the CM, and let the MOI about the axis passing through the CM be I_{CM}. Then, we have:

$$I = I_{\text{CM}} + md^2 \tag{7}$$

In very niche situations, you might require the converse: finding I_{CM} from I. However, the parallel axis theorem from Equation (7) still holds.

Example 1.3. Using the parallel axis theorem, find the MOI of a thin rod of mass m and length L about an axis through one end and perpendicular to its length. You might require the table of MOIs.

Solution. From the table, we know that the MOI of the thin rod about an axis through its centre and perpendicular to its length is $I_{\text{CM}} = \frac{1}{12}mL^2$. And, the distance between the CM (the centre of the rod) and the end of the rod is $d = \frac{L}{2}$. Hence, by the parallel axis theorem,

$$I = I_{\text{CM}} + md^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \frac{1}{3}mL^2$$

Remark. When using the parallel axis theorem, one of the rotational axes we are considering **must pass through the CM!** In other words, it is **wrong** to say $I_2 = I_1 + md^2$ for any two random axes!

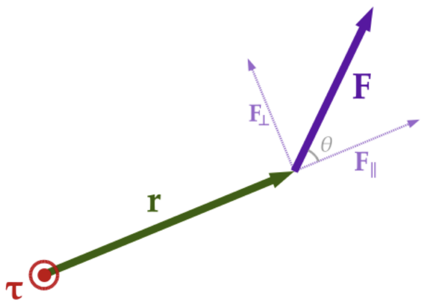
1.1.4 Torque

We shall slowly begin our discussion on rotational dynamics.

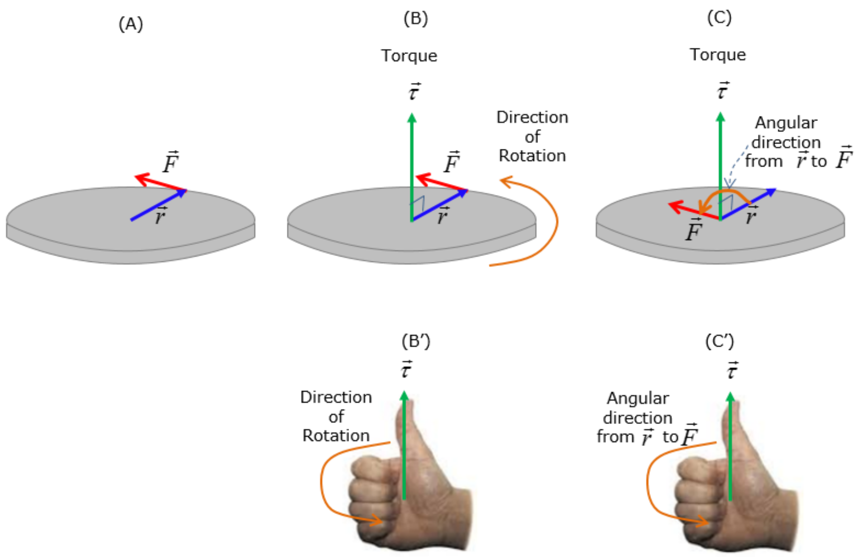
The **torque** measures the ability of a force to cause an angular acceleration. For a force $\boldsymbol{F}$ acting at a position of $\boldsymbol{r}$ from the origin, the magnitude of the torque $\boldsymbol{\tau}$ is defined as:

$$\tau = rF\sin\theta = rF_{\perp} \tag{8}$$

where θ is the angle between $\boldsymbol{F}$ and $\boldsymbol{r}$.



Note that the second equality is true because $F_{\perp} = F\sin\theta$. Since $\boldsymbol{\tau}$ is a **vector,** it still has a direction, which can be found using the **right-hand grip rule** as follows:



Most of the time, the rotations are usually clockwise or anti-clockwise.

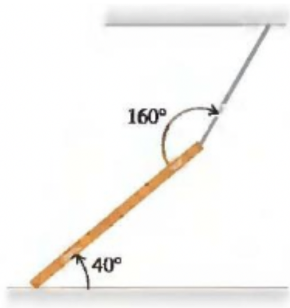
Essentially, torque is the rotational analogue of the force. When we dealt with static (translational) equilibrium, one of our conditions was $\boldsymbol{F}_{\text{net}} = \mathbf{0}$. Now that we have upgraded to considering rotation, the two conditions for equilibrium are:

$$\text{Equilibrium:} \quad \boldsymbol{F}_{\text{net}} = \mathbf{0} \quad \text{and} \quad \boldsymbol{\tau}_{\text{net}} = \mathbf{0} \tag{9}$$

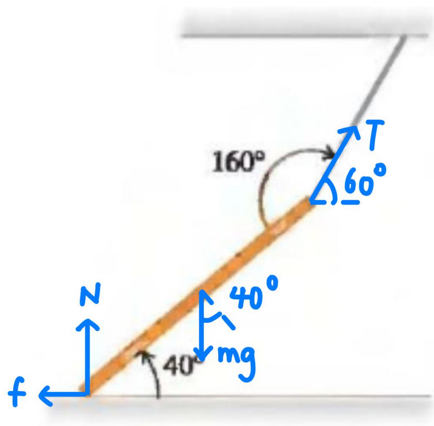
Do be careful about the direction of rotation when adding up the torques!

Remark. We usually want to compute the torque about a pivot point such that we can **eliminate as many unknown forces as possible.** Note that if a force acts at a point or acts through a point, it contributes no torque. Hence, this is a smart way to eliminate unknowns from our rotational equilibrium equation!

Example 1.4. A uniform 250 kg beam is supported by a cable connected to the ceiling, as shown in the diagram. The lower end of the beam rests on the floor. (a) What is the tension in the cable? (b) What is the minimum coefficient of static friction between the beam and the floor for the beam to remain in this position?



Solution. (a) Let's first draw the FBD of the beam:



Let's assume the beam has a length of L. Since the beam is translationally and rotationally stationary, using Equation (9), we have:

$$\text{Translational Equilibrium } (x\text{-direction}): \quad T\cos 60^\circ = f$$

$$\text{Translational Equilibrium } (y\text{-direction}): \quad T\sin 60^\circ + N = mg$$

$$\text{Rotational Equilibrium:} \quad (mg\cos 40^\circ)\left(\frac{L}{2}\right) = (T\sin 20^\circ)(L)$$

At this juncture, notice that we picked the pivot point as the left end of the stick (on the ground) for our rotational equilibrium equation. This is a good choice, because it allows us to eliminate the torques by N and f, which are unknown forces.

After some work (left as an exercise), you will get $T = 2750\,\text{N}$.

(b) At the extreme case, $f = \mu N$. Hence,

$$T\cos 60^\circ = \mu\left(mg - T\sin 60^\circ\right) \quad \Longrightarrow \quad \mu = \frac{T\cos 60^\circ}{mg - T\sin 60^\circ} = 19.4$$

(Note that the denominator is rather small, so rounding errors for T can contribute to large differences in numerical answers.)

1.1.5 N2L For Rotation

Since torque, MOI and angular acceleration are the rotational analogues of force, mass and acceleration respectively, this prompts us to think about whether we can write a rotational analogue of N2L.

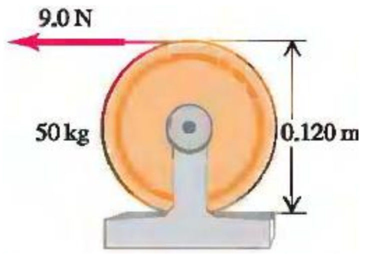
Indeed, we can! Just as $\boldsymbol{F}_{\text{net}} = m\boldsymbol{a}$, we have:

$$\boldsymbol{\tau}_{\text{net}} = I\boldsymbol{\alpha} \tag{10}$$

Observe that when $\boldsymbol{\alpha} = \mathbf{0}$ (i.e. rotational equilibrium), $\boldsymbol{\tau}_{\text{net}} = \mathbf{0}$, which makes sense as per Equation (9).

Remark. When evaluating $\boldsymbol{\tau}$ and I, we must ensure that we evaluate both of them through the **same rotational axis** for Equation (10) to work.

Example 1.5. Consider a wheel (a uniform disk) of mass 50 kg and diameter 0.120 m. It is pulled with a force of 9.0 N as shown in the diagram below. Ignore friction at the axle. (a) Find the magnitude of the tangential acceleration at a point on the rim of the wheel. (b) Find the magnitude and direction of the force exerted by the axle on the wheel.



Solution. (a) Let's take the pivot point as the centre of the wheel. This is a smart choice because the force exerted by the axle on the wheel contributes no torque (because it acts through this point), so we do not need to care about it yet.

The torque exerted by the external pulling force on the wheel is $\tau = Fr$. The MOI of the wheel about its centre is $I = \frac{1}{2}mr^2$. Hence, applying Equation (10), we can find the angular acceleration α of the wheel:

$$\tau = I\alpha \quad \Longrightarrow \quad Fr = \frac{1}{2}mr^2\alpha \quad \Longrightarrow \quad \alpha = \frac{2F}{mr}$$

Hence, the tangential acceleration at a point on the rim of the wheel is:

$$a = r\alpha = \frac{2F}{m} = \frac{2\,(9.0)}{50} = 0.36\,\text{m/s}^2$$

(b) We know that the wheel is in translational equilibrium, because it is not moving. Hence, the force exerted by the axle on the wheel must balance out the external pulling force and the weight of the wheel.

Since the external pulling force and the weight of the wheel and perpendicular, the magnitude of the force exerted by the axle is

$$F_{\text{axle}} = \sqrt{F^2 + (mg)^2} = \sqrt{(9.0)^2 + (50 \times 9.81)^2} = 491\,\text{N}$$

and the direction of the force exerted by the axle is

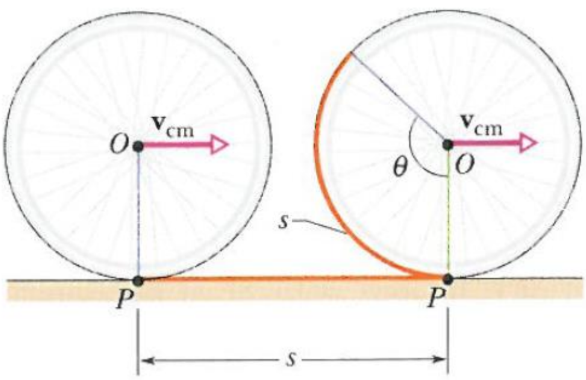
$$\theta = \tan^{-1}\left(\frac{mg}{F}\right) = \tan^{-1}\left(\frac{50 \times 9.81}{9.0}\right) = 88.9^\circ \text{ North of East}$$

1.1.6 Rolling

Rolling is a type of motion that combines both translation and rotation. When we say a ball is rolling across a table, it is both translating (moving forward) and rotating (spinning).

When we discuss rolling, we usually consider two regimes: **rolling without slipping** and **rolling with slipping.**

For an object that is **rolling without slipping,** when the centre of the object translates forward by some distance s, a point on the surface/rim of the object would have also moved through a distance s. It experiences **static friction** at its bottom, pointing **opposite** to the direction of motion to create the necessary torque for the object to roll in the desired direction.



This allows us to write the following relationships, known as the **non-slip conditions:**

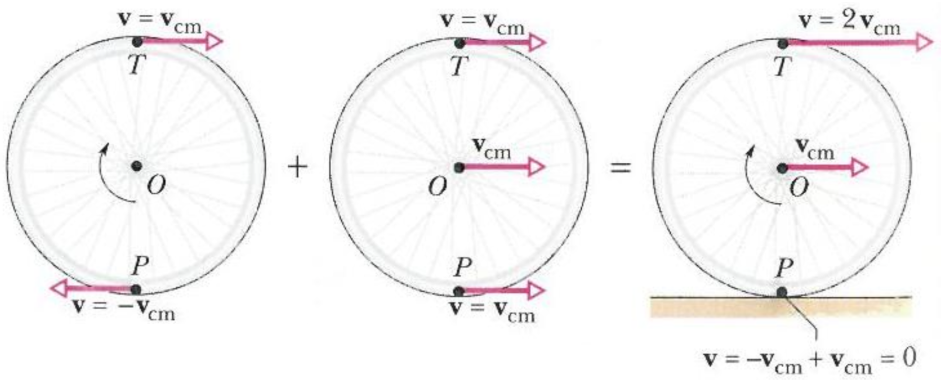
$$\text{Rolling Without Slipping:} \quad s = R\theta, \quad v_{\text{CM}} = R\omega, \quad a_{\text{CM}} = R\alpha \tag{11}$$

Using this knowledge, we can find the speeds of each point on the object:

$$\boldsymbol{v} = \boldsymbol{v}_{\text{CM}} + \boldsymbol{v}_{\text{rot}} \tag{12}$$

where $\boldsymbol{v}_{\text{rot}}$ arises due to rotation alone. (Note that this is a vector sum!)

As such, the velocity of the point touching the ground is 0, and the velocity of the point on the top is $2v_{\text{CM}}$, as illustrated by the diagram below:



For an object that is **rolling with slipping,** unfortunately, Equation (11) **does not hold!** There is no straightforward way to relate the linear and angular quantities. It experiences **kinetic friction** at its bottom, but do note that its direction depends on the direction of the velocity of the point at the bottom (it will oppose the velocity's direction).

In a sense, for rolling with slipping, you lose one equation (the non-slip condition), but you gain one equation (the kinetic friction equality compared to the static friction inequality).

Example 1.6. A wheel of radius R is moving to the right ($+x$-axis) at a speed of v_0 and is rolling without slipping. Find the velocity of the point at a distance R to the right of the centre.

Solution. Using Equation (12), we have $\boldsymbol{v}_{\text{CM}} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$ and $\boldsymbol{v}_{\text{rot}} = \begin{pmatrix} 0 \\ -R\omega \end{pmatrix} = \begin{pmatrix} 0 \\ -v_0 \end{pmatrix}$ because of the rolling without slipping condition $v_0 = R\omega$, and because this point is moving vertically downwards due to the clockwise rotation (note that the wheel has to rotate clockwise for it to be moving to the right without slipping).

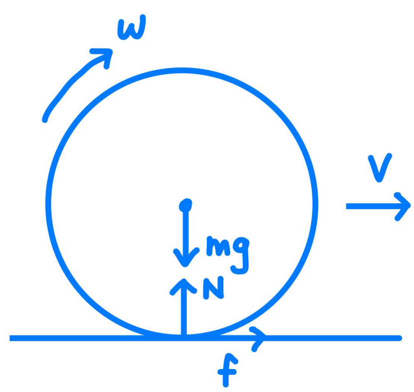
Hence, the velocity is

$$\boldsymbol{v} = \boldsymbol{v}_{\text{CM}} + \boldsymbol{v}_{\text{rot}} = \begin{pmatrix} v_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -v_0 \end{pmatrix} = \begin{pmatrix} v_0 \\ -v_0 \end{pmatrix}$$

so its magnitude is $\sqrt{2}v_0$ and its direction is 45° South of East.

Example 1.7. Consider a yo-yo (a uniform cylinder of mass m and radius R) that is initially rotating at an angular velocity of ω_0 clockwise, but not translating. At $t = 0$, it is put immediately on a rough horizontal surface with coefficient of friction μ. Find the time taken for the yo-yo to reach a state of rolling without slipping.

Solution. Qualitatively, what is happening is that the yo-yo slows down rotationally because the kinetic friction at the bottom opposes the clockwise rotation, but it speeds up translationally because the kinetic friction accelerates the yo-yo forward, as per the FBD below:



At some point, since ω decreases and v increases, we will reach a point whereby the non-slip condition $v = R\omega$ is achieved!

Writing N2L for the yo-yo,

$$f = ma \implies \mu mg = ma \implies a = \mu g$$

Writing the rotational analogue of N2L for the yo-yo,

$$\tau = I\alpha \implies -\mu mgR = \frac{1}{2}mR^2\alpha \implies \alpha = -\frac{2\mu g}{R}$$

Hence, the translational and rotational kinematics equations give:

$$v = at = \mu gt$$

$$\omega = \omega_0 + \alpha t = \omega_0 - \frac{2\mu gt}{R}$$

Finally, using the non-slip condition, we can get the time needed:

$$v = R\omega \implies \mu gt = R\omega_0 - 2\mu gt \implies t = \frac{R\omega_0}{3\mu g}$$

1.1.7 Energy In Rotation

Since MOI and angular velocity are the rotational analogues of mass and velocity respectively, we can also write a rotational analogue of kinetic energy. This is termed **rotational kinetic energy,** and is given by

$$K_{\text{rot}} = \frac{1}{2}I\omega^2 \tag{13}$$

To be precise from now on, we will call $K_{\text{trans}} = \frac{1}{2}mv^2$ the **translational kinetic energy.**

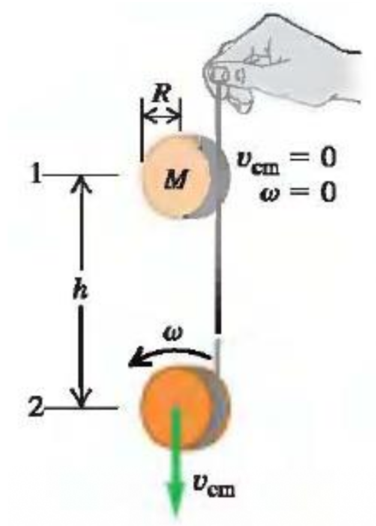
Since rolling is a combination of translation and rotation, rolling objects possess both types of kinetic energy. Their total kinetic energy is given by

$$K = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2}m{v_{\text{CM}}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2 \tag{14}$$

Notice that this implies that we can find the total kinetic energy by finding the translational kinetic energy by treating the object as a point mass located at the CM, and then adding the rotational kinetic energy of the object about the CM.

Remark. Note that in Equation (14), both v and I are evaluated about the CM. Always just stick to the CM for calculations to not confuse yourself!

Example 1.8. A primitive yo-yo is made by wrapping a string several times around a solid cylinder of mass M and radius R. You hold the end of the string stationary while releasing the cylinder with no initial motion. The string unwinds but does not slip or stretch as the cylinder drops and rotates. (a) Determine the CM speed v_{CM} and the angular speed ω of the cylinder after it has dropped a distance h, **using energy methods.** (b) Explain qualitatively what will happen to v_{CM} and ω if we replace the solid cylinder with a hollow one of the same mass M and radius R.



Solution. (a) This is just a simple application of COE, taking into account the rotational KE. Initially, the cylinder has GPE, and it is converted into translational + rotational KE.

Hence, by COE, we have:

$$Mgh = \frac{1}{2}M{v_{\text{CM}}}^2 + \frac{1}{2}I_{\text{CM}}\omega^2 = \frac{1}{2}M{v_{\text{CM}}}^2 + \frac{1}{4}MR^2\omega^2$$

Using the non-slip condition $v_{\text{CM}} = R\omega$, we then have:

$$Mgh = \frac{1}{2}M{v_{\text{CM}}}^2 + \frac{1}{4}M{v_{\text{CM}}}^2 = \frac{3}{4}M{v_{\text{CM}}}^2 \quad \Longrightarrow \quad v_{\text{CM}} = \sqrt{\frac{4}{3}gh}, \quad \omega = \frac{1}{R}\sqrt{\frac{4}{3}gh}$$

Of course, you could have also solved this using torque (and you should try it for yourself!), but energy methods tend to be faster, especially when COE holds.

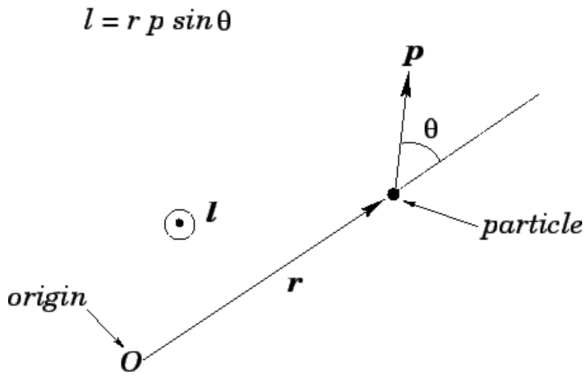
(b) The hollow cylinder of mass M and radius R has $I_{\text{CM}} = MR^2$, hence the MOI about the CM increased. Qualitatively, this means a larger proportion of the total KE will be associated with rotational KE, leaving less translational KE. Hence, v_{CM} and ω will decrease.

1.1.8 Angular Momentum

The **angular momentum** is the rotational analogue of the linear momentum. For a mass m moving at velocity $\boldsymbol{v}$ at a position $\boldsymbol{r}$ from the origin, the magnitude of the angular momentum $\boldsymbol{L}$ is defined as:

$$L = mvr\sin\theta = pr\sin\theta \tag{15}$$

where $p = mv$ is the magnitude of the linear momentum, and θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{r}$.



Since $\boldsymbol{L}$ is a **vector,** it still has a direction, which can be found using the **right-hand grip rule**, like what we did for torque.

If we further consider a rigid body with moment of inertia I about some axis, and consider that it is rotating about that axis with angular velocity ω, then the magnitude of the angular momentum $\boldsymbol{L}$ is:

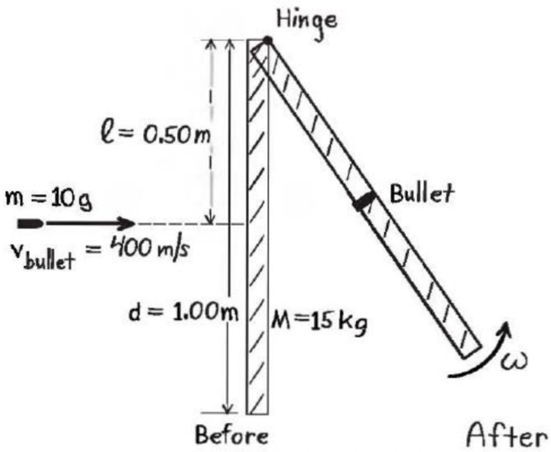
$$L = I\omega \tag{16}$$

Since torque and angular momentum are the rotational analogues of force and linear momentum respectively, and since we know $\boldsymbol{F}_{\text{net}} = \frac{\Delta \boldsymbol{p}}{\Delta t}$, hence we have:

$$\boldsymbol{\tau}_{\text{net}} = \frac{\Delta \boldsymbol{L}}{\Delta t} \tag{17}$$

From Equation (17), we can see that when a system has **no net external torque** acting on it (with respect to some rotational axis), then the **total angular momentum is conserved** (with respect to the same rotational axis). This is what we call the **conservation of angular momentum,** or COAM for short.

Example 1.9. A door that is 1.00 m wide and has a mass of 15 kg is hinged at one side so that it can rotate without friction about a vertical axis. A police officer fires a bullet with a mass of 10 g and a speed of 400 m/s into the exact centre of the door, in a direction perpendicular to the plane of the door. (a) Find the angular speed of the door just after the bullet embeds itself in the door. (b) Find the change in kinetic energy from this process.



Solution. (a) The rotational axis in this case is the axis of the hinge. This is a good choice of axis, because it allows us to utilise COAM. Physically, the hinge exerts a force on the door as it absorbs the impact of the bullet, but by choosing this as our rotational axis, we can neglect its associated torque, and hence $\boldsymbol{\tau}_{\text{net}} = \boldsymbol{0}$.

Hence, using COAM, we equate the initial angular momentum of the bullet to the final angular momentum of the bullet-door system:

$$\frac{m_{\text{b}} v_{\text{b}} d}{2} = I_{\text{d}}\omega + \frac{m_{\text{b}}\omega d^2}{4} = \frac{1}{3}Md^2\omega + \frac{m_{\text{b}}\omega d^2}{4}$$

where we need to be careful that $I_{\text{d}} = \frac{1}{3}Md^2$ for the MOI of the door, since we are evaluating it about the hinge.

Hence, we can solve for ω:

$$\omega = \frac{\frac{m_\text{b} v_\text{b} d}{2}}{\frac{1}{3}Md^2 + \frac{m_\text{b} d^2}{4}} = \frac{\frac{1}{2}(0.010)(400)(1.00)}{\frac{1}{3}(15)(1.00)^2 + \frac{1}{4}(0.010)(1.00)^2} = 0.40\,\text{rad/s}$$

(b) The initial KE is just the translational KE of the bullet, while the final KE is the translational + rotational KE of the bullet-door system. Hence, the change in KE is given by:

$$\Delta K = K_\text{f} - K_\text{i} = \frac{1}{2}I_\text{d}\omega^2 + \frac{1}{2}m_\text{b}\left(\frac{\omega d}{2}\right)^2 - \frac{1}{2}m_\text{b}v_\text{b}^2$$

$$= \frac{1}{2}\left(\frac{1}{3}\right)(15)(1.00)^2(0.40)^2 + \frac{1}{2}(0.010)\left(\frac{(0.40)(1.00)}{2}\right)^2 - \frac{1}{2}(0.010)(400)^2 = -799.6\,\text{J}$$

Energy is lost because the collision is completely inelastic, since the bullet embedded itself into the door. Some might also have been dissipated into heat and sound.

1.2 Simple Harmonic Motion

Simple harmonic motion (SHM) is a special type of periodic motion where a body oscillates in a sinusoidal fashion about equilibrium. SHM is typically one of the *most important* types of oscillations you will encounter in pretty much every topic of physics.

1.2.1 Definitions and Terminology

The equation of motion for a simple harmonic oscillator (an object which undergoes SHM) in 1 dimension is defined to be

$$a = -\omega^2 x \tag{18}$$

where a is the acceleration, x is the displacement from equilibrium and ω is a constant known as the **angular frequency** with SI unit radians per second (rad s^{-1}), whose significance we shall explain shortly. Here, the negative sign indicates that the acceleration is *opposite* in direction to the displacement, so that the object always accelerates towards equilibrium (here the origin).

Note: This equation also applies for any displacement-like and acceleration-like quantities, for example angle θ and angular acceleration α respectively.

It turns out that the general solution to this equation of motion is sinusoidal motion,

$$x(t) = A\sin(\omega t + \phi) \tag{19}$$

where A is the maximum displacement from equilibrium called the **amplitude**, ω is the angular frequency and ϕ is the initial value of the **phase** φ, a quantity that describes the progress of the object through its cycle with SI unit radians (rad).

From this equation, we can see that since the sine function is periodic, the motion of the object is also periodic, i.e. repeats itself. The motion repeats after a duration called the **period** T, related to the angular frequency by

$$\omega T = 2\pi \quad \implies \quad \omega = \frac{2\pi}{T} \tag{20}$$

as concluded by the solution to the equation of motion. We also sometimes encounter the term **frequency** f, the number of cycles per unit time, defined by

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{21}$$

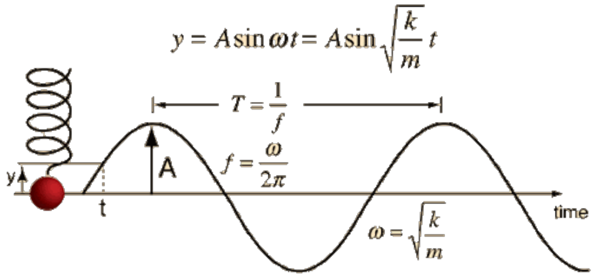
with SI unit Hertz (Hz $= \text{s}^{-1}$).

Note: Usually, physicists use the term "frequency" to mean the angular frequency, so it is important to check the units or the symbol used to distinguish between f and ω.

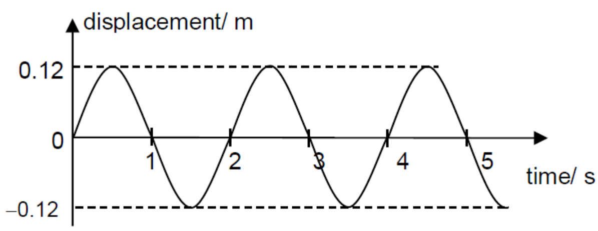
Observe that the amplitude and frequency are independent, i.e. for SHM, the period does not depend on the amplitude.

In summary, we have the following terminology used when describing SHM:

Term	**Definition**
Equilibrium	The position at which the net force is zero
Amplitude (A)	The maximum magnitude of displacement from equilibrium
Phase (φ)	The progress through the cycle, a full cycle being 2π
Period (T)	The time taken for one complete oscillation cycle
Frequency (f)	The number of complete oscillation cycles per unit time
Angular frequency (ω)	The change in the phase per unit time



Example 1.10. A pendulum oscillates in simple harmonic motion with its displacement from the equilibrium position graphed against time as shown below. Determine (a) the angular frequency, (b) the equation for displacement in terms of time, (c) the maximum acceleration, (d) the position(s) where the speed is maximum.



Solution. (a) The period is the duration for one cycle (one peak and one trough), i.e. $T = 2$ s, so the angular frequency is $\omega = \frac{2\pi}{T} = \pi$ rad/s.

(b) The amplitude is the maximum displacement, $A = 0.12$ m. From the graph we can infer that $x(t) = A \sin \omega t = 0.12 \sin \pi t$ where x is in metres and t is in seconds.

(c) By the definition of SHM, $a_{\text{max}} = \omega^2 x_{\text{max}} = \omega^2 A = 0.12\pi^2 \approx 1.18$ m/s^2.

(d) The speed is maximum when the acceleration is zero, which clearly occurs at the equilibrium position.

1.2.2 Oscillations with Force Analysis

Two common examples of mechanical systems that exhibit SHM are spring-mass systems and pendulums. We can determine the angular frequency, and so the period, of the oscillations by analysing the system physically.

For example, for a spring-mass system, Hooke's law and Newton's 2$^{\text{nd}}$ law give

$$F_{\text{net}} = ma = -kx \quad \implies \quad a = -\frac{k}{m}x$$

i.e. comparing with the definition of SHM, the angular frequency and period for a spring-mass system is

$$\omega = \sqrt{\frac{k}{m}} \quad \implies \quad T = 2\pi\sqrt{\frac{m}{k}} \tag{22}$$

where k is the spring constant of the spring and m is the mass.

On the other hand, for a (simple) pendulum, analysing the gravitational torque about the pivot gives

$$\tau_{\text{net}} = I\alpha = m\ell^2\alpha = mg\sin\theta \quad \implies \quad \alpha = \frac{g}{\ell}\sin\theta$$

Notice that this does not exactly fit the definition for SHM. However, if we assume that the maximum angle (i.e. amplitude) of the pendulum is *small*, we can take the small-angle approximation $\sin\theta \approx \theta$, reducing the equation of motion to the definition of SHM. This gives us that the angular frequency and period for a **small-angle** pendulum is

$$\omega = \sqrt{\frac{g}{\ell}} \quad \implies \quad T = 2\pi\sqrt{\frac{\ell}{g}} \tag{23}$$

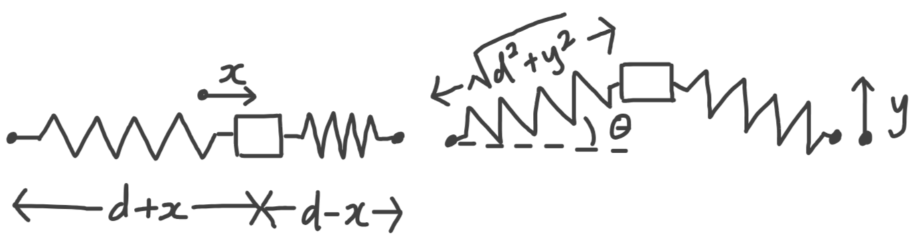
where g is the gravitational acceleration and ℓ is the length of the pendulum.

Note: This is only an approximation; the actual period deviates from this value for large amplitudes! The motion of the pendulum is indeed approximately SHM for small angles, but it deviates from ideal SHM for larger angles.

Note: The small-angle approximation $\sin\theta \approx \theta$ is useful to know, but not strictly required for SJPO.

Let us consider another example where we are asked to determine the frequency of oscillations of a physical system, which we shall analyse using Newton's 2$^{\text{nd}}$ law.

Example 1.11. A mass m at rest at the origin is connected to two identical springs of spring constant k and zero rest length fixed at the points $(\pm d, 0)$. Determine the frequency of oscillations when the mass is given a small displacement in the (a) x-direction, (b) y-direction.



Solution. (a) When the mass is displaced to coordinate x, the lengths of the two springs are $d + x$ and $d - x$ respectively, so using Hooke's law and analysing the directions of the forces,

$$F_{\text{net}} = ma = -k(d + x) + k(d - x) = -2kx \quad \implies \quad a = -\frac{2k}{m}x$$

Therefore, by the definition of SHM, the frequency of oscillations is

$$\omega = \sqrt{\frac{2k}{m}}$$

(b) Now, when the mass is displaced to coordinate y, the lengths to the two springs are both $\sqrt{d^2+y^2}$. However, the spring forces do not act parallel to the y-direction. Due to symmetry, the x-components cancel out, so we take the y-components of the spring forces to get

$$F_{\text{net}} = -2k\sqrt{d^2+y^2} \cdot \frac{y}{\sqrt{d^2+y^2}} = -2ky \quad \Longrightarrow \quad a = -\frac{2k}{m}y$$

which implies that the frequency of oscillations when displaced in the y-direction is the same as if it were displaced in the x-direction,

$$\omega = \sqrt{\frac{2k}{m}}$$

1.2.3 Relationships between Quantities

By differentiating the equation of motion with respect to time, we can obtain that the velocity and acceleration of the object in SHM is

$$x(t) = A\sin(\omega t + \phi) \tag{24}$$
$$v(t) = \omega A\cos(\omega t + \phi) \tag{25}$$
$$a(t) = -\omega^2 A\sin(\omega t + \phi) \tag{26}$$

Notice that $a = -\omega^2 x$ as expected. We can also derive a useful relation between x and v by using the Pythagorean trigonometric identity $\sin^2\theta + \cos^2\theta = 1$, giving us

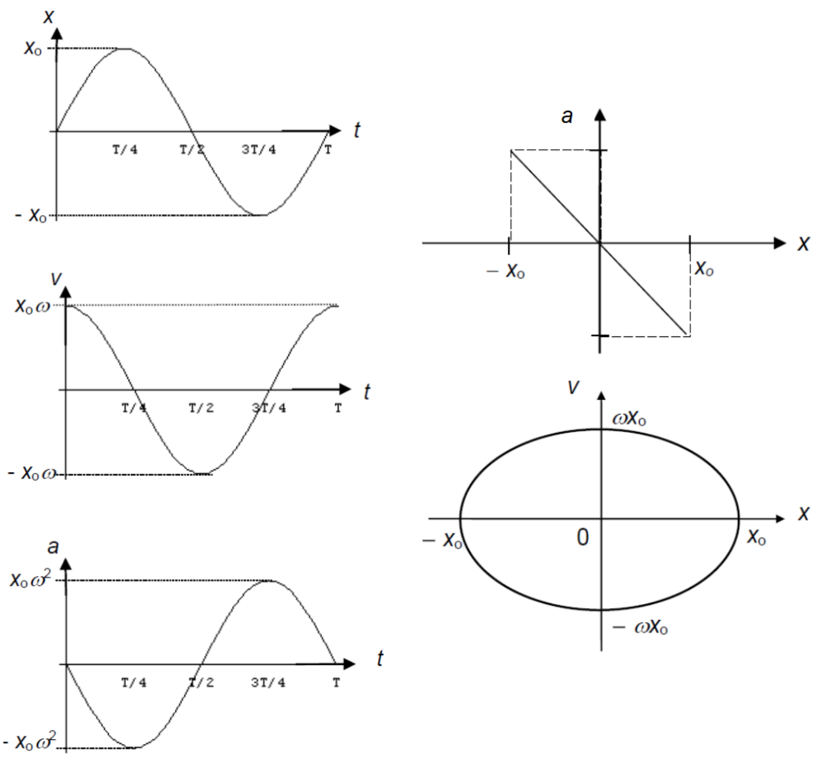
$$\left(\frac{x}{A}\right)^2 + \left(\frac{v}{\omega A}\right)^2 = 1 \quad \Longrightarrow \quad v^2 = \omega^2\left(A^2 - x^2\right) \tag{27}$$

This implies that the maximum speed of an object in SHM is

$$v_{\max} = \omega A \tag{28}$$

occurring at $x = 0$, i.e. at the position of equilibrium.

We can plot the graphs of these quantities against one another to understand their relationships:



As observed,

- the graphs of x, v and a against t are sinusoidal,
- the graphs of x and v against t are 90° apart, and the graphs of v and a against t are 90° apart,
- the graphs of x and a against t are 180° apart (as expected by definition),
- the graph of a against x is a straight line through the origin with *negative gradient* (as expected by definition),
- the graph of v against x is an *ellipse*, with semi-major axis A and semi-minor axis $v_{\max}$.

Example 1.12. An object moving in simple harmonic motion on the x-axis has an amplitude of $A = 0.02$ m, a frequency of $f = 20$ Hz and starts from rest at its maximum displacement in the positive direction at time $t = 0$. Determine (a) the period T of oscillations, (b) its acceleration at $t = \frac{T}{8}$, (c) its speed at $x = -\frac{A}{2}$.

Solution. (a) By definition, $T = \frac{1}{f} = 0.05$ s. (Here the frequency is given in Hertz with the symbol f, so we know that this "frequency" is not the angular frequency!)

(b) Since the object starts from rest at $x = A$, we must have $x(t) = A\cos\omega t$, so the position of the object at $t = \frac{T}{8}$ is

$$x = A\cos(\omega t) = A\cos\left(\frac{2\pi}{T}\cdot\frac{T}{8}\right) = \frac{A}{\sqrt{2}}$$

Therefore, by the definition of SHM, the acceleration is

$$a = -\omega^2 x = -(2\pi f)^2\frac{A}{\sqrt{2}} = -2\sqrt{2}\pi^2 f^2 A \approx -223 \text{ m/s}^2$$

(c) Since we are given the position, we can use the relation between velocity and position to get

$$v^2 = \omega^2\left(A^2 - x^2\right) = \omega^2\left(A^2 - \left(-\frac{A}{2}\right)^2\right) = \frac{3}{4}\omega^2 A^2$$

We are only asked for the speed $|v|$, so we safely take the square root to get

$$|v| = \sqrt{\frac{3}{4}(2\pi f)^2 A^2} = \sqrt{3}\pi f A \approx 2.18 \text{ m/s}$$

1.2.4 Energy

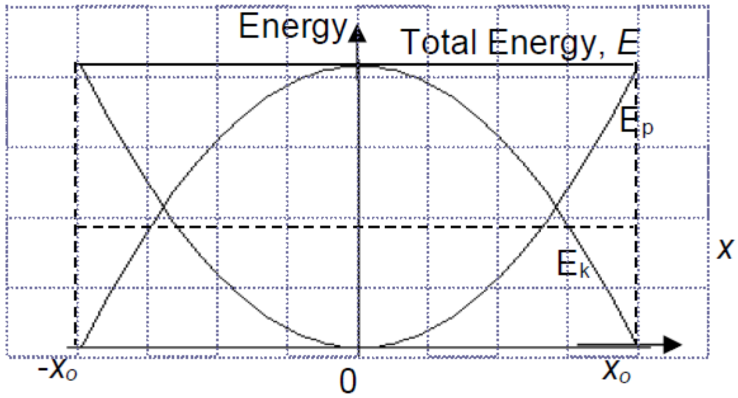
Let us return to the example of a spring-mass system, with a spring of spring constant k and a mass m. The total energy E of the mass is conserved, and is comprised of the kinetic energy K and the elastic potential energy U. We can calculate these quantities explicitly using our SHM formulas to get

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) \tag{29}$$

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2 \tag{30}$$

$$E = K + U = \frac{1}{2}m\omega^2 A^2 \tag{31}$$

where we used Equation (22) to express k in terms of m and ω. We can plot these quantities against x to get a sense of how the energy varies with position:



Notice that energy is converted from kinetic to potential energy as the object travels away from equilibrium and vice versa, occurring every half cycle. We can also rewrite the total energy as

$$E = \frac{1}{2}kA^2 = \frac{1}{2}mv_{\text{max}}^2 \tag{32}$$

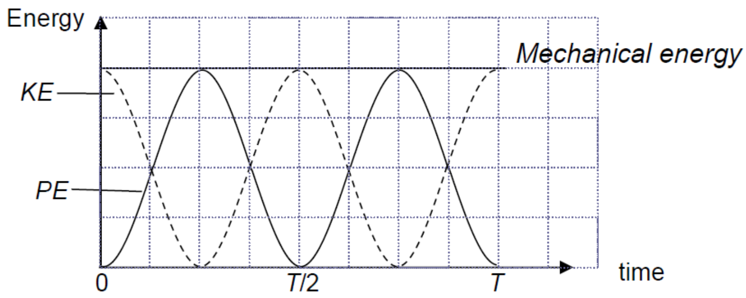
which are the maximum potential energy U_{max} and kinetic energy K_{max} respectively, and also implies that the total energy is proportional to the square of the amplitude.

We can also express the potential and kinetic energies in terms of time:

$$K = \frac{1}{2}m\omega^2 A^2 \cos^2(\omega t + \phi) \tag{33}$$

$$U = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \tag{34}$$

and now it should be easy to see that the total energy is constant, due to the Pythagorean trigonometric identity. We can also plot these quantities against t to get a sense of how the energy varies with time:



We can see that energy is in fact converted between potential and kinetic energy every half cycle.

Example 1.13. A mass $m = 0.5$ kg connected to a light spring with a spring constant $k = 20$ N/m oscillates on a frictionless horizontal surface with amplitude $A = 3.0$ cm. (a) Calculate the total energy of the system. (b) Calculate the kinetic and potential energy of the system when the displacement is $x = 2.0$ cm.

Solution. (a) Since the total energy is conserved, we can calculate it from the elastic potential energy at its amplitude (where it has no kinetic energy), which gives

$$E = \frac{1}{2}kA^2 = 9.0 \times 10^{-3} \text{ J}$$

(b) The potential energy is easier to calculate, being

$$U = \frac{1}{2}kx^2 = 4.0 \times 10^{-3} \text{ J}$$

The kinetic energy and the potential energy must then sum to the total energy which we have calculated earlier, so

$$K = E - U = 5.0 \times 10^{-3} \text{ J}$$

1.2.5 Oscillations with Energy Analysis

One thing of note is that this analysis does not just apply to spring-mass systems — in fact, **any system with potential energy proportional to the square of the displacement will undergo SHM!** This gives us an alternative way to derive the frequency of oscillations by using energy.

For example, consider the *small-angle* pendulum again. Here, we shall analyse the system using the angular displacement θ and the angular velocity Ω (capital to avoid conflict with the frequency ω), so total energy of the system can be expressed in terms of these variables as

$$E = \frac{1}{2}k_{\text{eff}}\theta^2 + \frac{1}{2}m_{\text{eff}}\Omega^2$$

where k_{eff} is the effective spring constant and m_{eff} is the effective mass. (Do note that these quantities do not necessarily have their typical units, since we are using angular quantities!)

The potential energy for this system is the gravitational potential energy, which is

$$U = mg\ell(1 - \cos\theta)$$

For small angles, we can apply the small-angle approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$ to obtain

$$U \approx \frac{1}{2}mg\ell\theta^2$$

which is directly proportional to θ^2! This implies that the pendulum will undergo SHM. Then, the kinetic energy for this system is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\ell^2\Omega^2$$

Therefore, the total energy is

$$E = \frac{1}{2}mg\ell\theta^2 + \frac{1}{2}m\ell^2\Omega^2$$

which implies that the effective spring constant is $k_{\text{eff}} = mg\ell$ and the effective mass is $m_{\text{eff}} = m\ell^2$, i.e. the oscillation frequency is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{g}{\ell}}$$

exactly as we derived previously using force analysis!

Note: The small-angle approximation $\cos\theta \approx 1 - \frac{\theta^2}{2}$ is also useful to know, but not strictly required for SJPO.

Let us consider another example where energy analysis may be more useful compared to force analysis.

Example 1.14. Two masses M and m connected by a spring of spring constant k are both free to move along the x-axis. Determine the frequency of oscillations of the system.

Solution. Firstly, realise that we can consider the system in the inertial centre of mass frame (since the frequency of oscillations does not change). Denote the velocities of masses M and m as V and v. By conservation of momentum, we have

$$mv + MV = 0 \qquad \Longrightarrow \qquad V = -\frac{m}{M}v$$

Now, we shall take our coordinate of interest to be the length x of the spring. In this case, our corresponding velocity is then the *relative velocity* between the two masses,

$$v_{\text{rel}} = v - V = \frac{M+m}{M}v$$

Our goal is then to express the energy of the system in terms of x and v_{rel}. The potential energy of the system is simply the elastic potential energy of the spring,

$$U = \frac{1}{2}kx^2$$

The kinetic energy of the system is then the total kinetic energy of both masses,

$$K = \frac{1}{2}mv^2 + \frac{1}{2}MV^2 = \frac{1}{2}\frac{m(M+m)}{M}v^2 = \frac{1}{2}\frac{Mm}{M+m}v_{\text{rel}}^2$$

Therefore, the total energy is

$$E = \frac{1}{2}kx^2 + \frac{1}{2}\frac{Mm}{M+m}v_{\text{rel}}^2$$

which implies that the effective spring constant is $k_{\text{eff}} = k$ and the effective mass is $m_{\text{eff}} = \frac{Mm}{M+m}$, i.e. the oscillation frequency is

$$\omega = \sqrt{\frac{k_{\text{eff}}}{m_{\text{eff}}}} = \sqrt{\frac{k(M+m)}{Mm}}$$

Note: The effective mass m_{eff} actually has significance here! This is called the **reduced mass**, often denoted by μ, and it allows us to effectively replace the two-body problem with an equivalent one-body problem with one of the masses stationary.

2 Problems

Problem 2.1 (SJPO 2011)**.** A drum has a radius of 0.40 m and a moment of inertia of 4.5 kg m^2. The frictional torque of the drum axle is 3.0 Nm. A 15 m length of rope is wound around the rim. The drum is initially at rest. A constant force is applied to the free end of the rope until the rope is completely unwound and slips off. At that instant, the angular velocity of the drum is 13 rad/s. The drum then decelerates and comes to a halt. Which of the following is the constant force applied to the rope closest to?

(A) 7.5 N

(B) 14 N

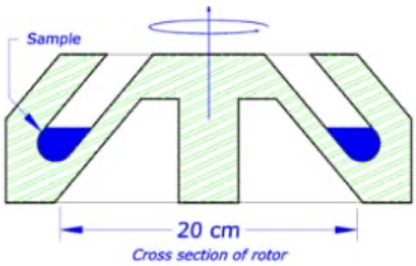
(C) 18 N

(D) 27 N

(E) 33 N

Solution. (E)

Problem 2.2 (SJPO 2013). Four samples of a colloidal aqueous mixture, each weighing 12.0 g, are placed in the rotor of a high speed centrifuge, equally spaced around the circumference of the rotor. The samples are located 10 cm from the axis of rotation of the rotor. If the centrifuge motor delivers a constant torque of 0.25 Nm and the empty rotor has a moment of inertia of 0.060 kg m^2, how long does it take for the rotor to accelerate to its operating state of 18000 rpm (rotations per minute)? You may neglect the change in position of the sample during acceleration.



(A) 7.6 s

(B) 456 s

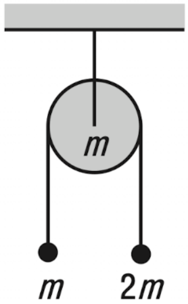
(C) 453 s

(D) 7.3 s

(E) 435 s

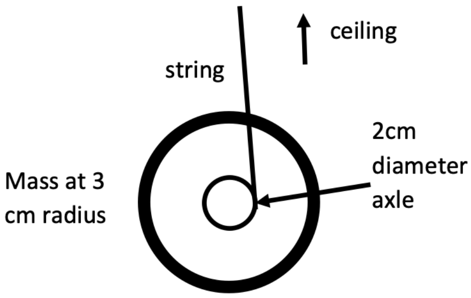
Solution. (B)

Problem 2.3. Consider a pulley system shown in the diagram below. The masses are m and $2m$, and the pulley is a uniform disk of mass m and radius r. The string is massless and does not slip with respect to the pulley. Find the acceleration of the masses.



Solution. $\frac{2}{7}g$

Problem 2.4 (SJPO 2016). A circular disc with an axle of diameter 2 cm, is attached with strings to the ceiling. The disc is rotated so that the strings wind up along the axle so that the disc is raised up to the ceiling. The string is long such that when the disc is released from rest, its centre of mass falls 2.0 m. The disc does not slip from the string. Assume the axle is massless and the disc has all of its 5 kg mass at radius 3 cm. Find the acceleration of the centre of mass of the disc near the bottom of the fall.



(A) g

(B) $\frac{2}{3}g$

(C) $\frac{1}{3}g$

(D) $\frac{1}{5}g$

(E) $\frac{1}{10}g$

Solution. (E)

Problem 2.5 ($F = ma$ 2019). Consider a flat uniform square of mass M and side length L. Cut a circle out of the square that has a diameter equal to the length of the side of the square, with the same centre as the square. Determine the moment of inertia of the remaining shape about an axis through the centre and perpendicular to the plane of the square.

(A) $\left(\frac{1}{6} - \frac{\pi}{32}\right) ML^2$

(B) $\left(\frac{1}{12} - \frac{\pi}{64}\right) ML^2$

(C) $\left(\frac{\pi}{24} - \frac{1}{3\pi}\right) ML^2$

(D) $\left(\frac{1}{2\pi} - \frac{1}{16}\right) ML^2$

(E) $\left(\frac{1}{2\pi} - \frac{1}{8}\right) ML^2$

Solution. (A)

Problem 2.6 ($F = ma$ 2011). A hollow cylinder with a very thin wall (like a toilet paper tube) and a block are placed at rest at the top of a plane with inclination θ above the horizontal. The cylinder rolls down the plane without slipping and the block slides down the plane. It is found that both objects reach the bottom of the plane simultaneously. What is the coefficient of kinetic friction between the block and the plane?

(A) 0

(B) $\frac{1}{3}\tan\theta$

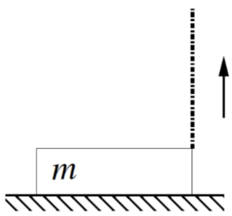
(C) $\frac{1}{2}\tan\theta$

(D) $\frac{2}{3}\tan\theta$

(E) $\tan\theta$

Solution. (C)

Problem 2.7 ($F = ma$ 2018). A uniform stick with mass m is originally on a horizontal surface. One end is attached to a vertical rope, which pulls up with a constant tension force F so that the centre of mass of the stick moves upward with acceleration $a < g$. What condition must the normal force N of the ground on the other end of the stick shortly after the right end of the stick leaves the surface satisfy?



(A) $N = mg$

(B) $mg > N > \frac{1}{2}mg$

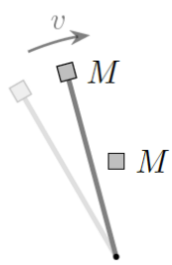
(C) $N = \frac{1}{2}mg$

(D) $\frac{1}{2}mg > N > 0$

(E) $N = 0$

Solution. (B) Hint: Torque balance when the right end just lifts off

Problem 2.8 ($F = ma$ 2020). A massless rigid rod is pivoted at one end, and a mass M is at the other end. Originally, the rod rotates frictionlessly about the pivot with a uniform angular velocity such that the mass M has speed v. The rotating rod collides with another mass M at its midpoint, which then sticks to the rod. After the collision, what is the kinetic energy of the system?



(A) $\frac{1}{4}Mv^2$

(B) $\frac{1}{3}Mv^2$

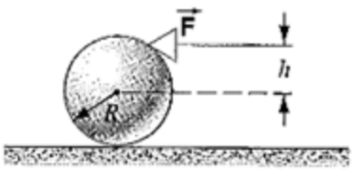
(C) $\frac{7}{18}Mv^2$

(D) $\frac{2}{5}Mv^2$

(E) $\frac{1}{2}Mv^2$

Solution. (D)

Problem 2.9. A billiard ball, initially at rest, is given a sharp impulse by a cue. The cue is held horizontally a distance h above the centre, as shown in the figure below. The ball leaves the cue with a speed v, and because of its "forward English", eventually acquires a final speed of $\frac{9}{7}v$. Find the required distance h to achieve this.



Solution. $\frac{4}{5}r$

Problem 2.10 ($F = ma$ 2021). A solid sphere sits at the top of a ramp of height h inclined at angle θ to the horizontal. Both the static and kinetic coefficients of friction between the sphere and the incline are $\mu_k = \mu_s = 0.2$. The sphere is released from rest at the top of the incline. For which of the following values of θ is the total kinetic energy of the sphere greatest when it reaches the bottom of the incline?

(A) 10°

(B) 45°

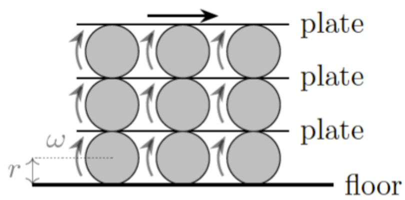
(C) 60°

(D) 80°

(E) It is the same for all of the choices above.

Solution. (A)

Problem 2.11 ($F = ma$ 2020). A system of cylinders and plates is set up as shown. The cylinders all have radius r, and roll without slipping to the right with an angular velocity ω. What is the speed of the top plate?



(A) ωr

(B) $2\omega r$

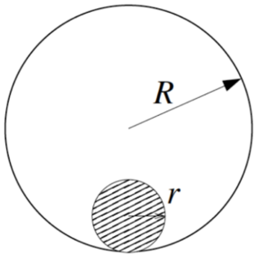
(C) $3\omega r$

(D) $4\omega r$

(E) $6\omega r$

Solution. (E)

Problem 2.12 ($F = ma$ 2018). A disc of radius r rolls uniformly without slipping around the inside of a fixed hoop of radius R. If the period of the disc's motion around the hoop is T, what is the instantaneous speed of the point on the disc opposite to the point of contact?



(A) $\frac{2\pi(R+r)}{T}$

(B) $\frac{2\pi(R+2r)}{T}$

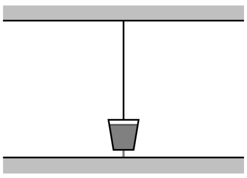
(C) $\frac{4\pi(R-2r)}{T}$

(D) $\frac{4\pi(R-r)}{T}$

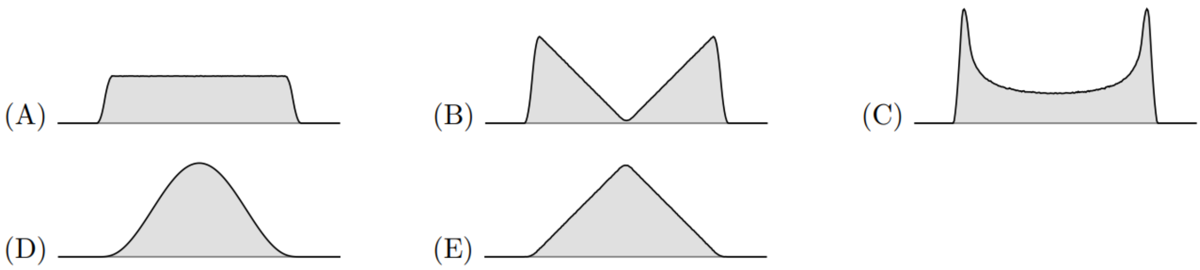
(E) $\frac{4\pi(R+r)}{T}$

Solution. (D)

Problem 2.13 ($F = ma$ 2024). A pendulum is made with a string and a bucket full of water. When the string is vertical, the bottom of the bucket is near the ground.

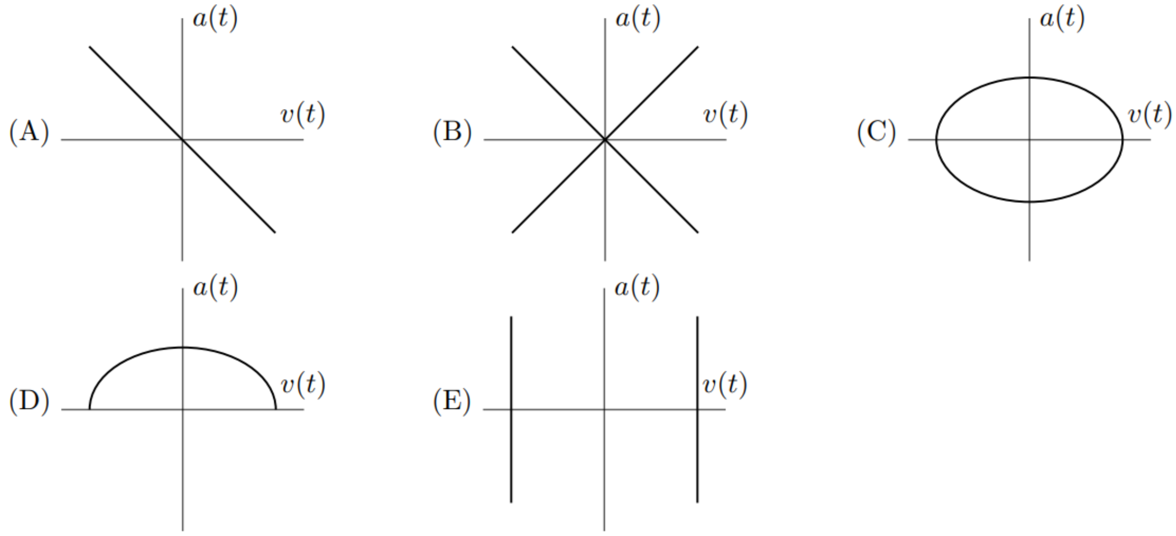


Then, the pendulum is set swinging with a small amplitude, and a very small hole is opened at the bottom of the bucket, which leaks water at a constant rate. After a few full swings, which of the following best shows the amount of water that has landed on the ground as a function of position?



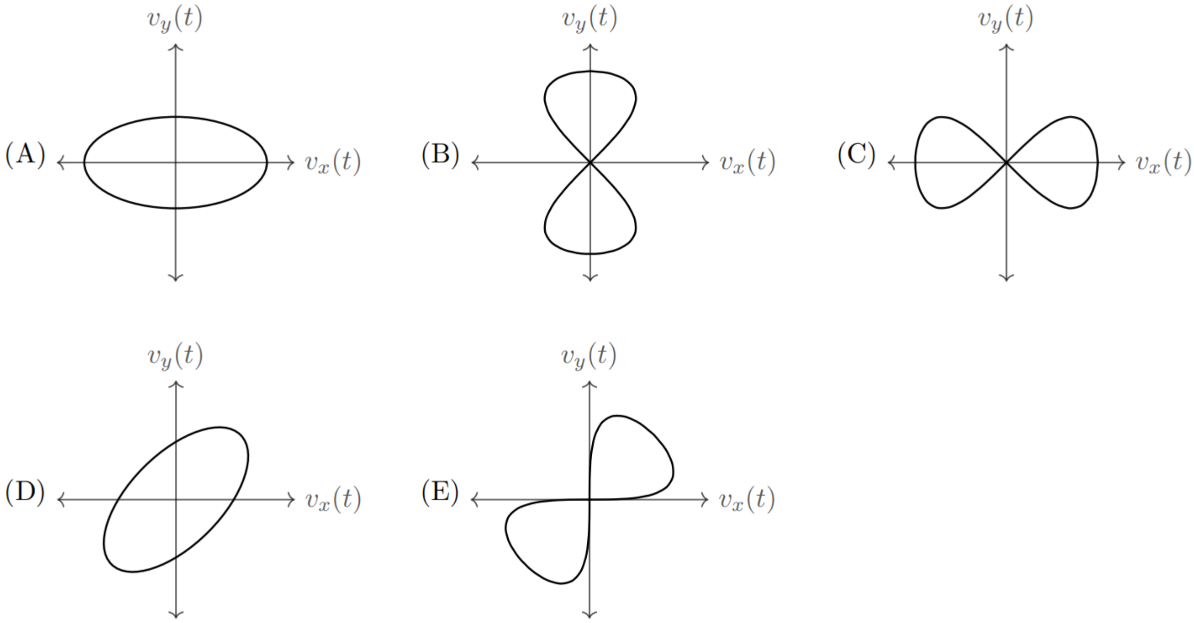
Solution. (C)

Problem 2.14 ($F = ma$ 2022). A mass attached to a spring is performing simple harmonic motion, with velocity $v(t)$ and acceleration $a(t)$. Which of the following could be a graph of the curve $(v(t), a(t))$ over a complete oscillation?



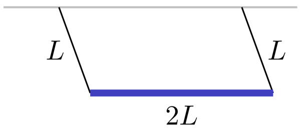
Solution. (C)

Problem 2.15 ($F = ma$ 2023). A mass attached to the end of a string oscillates like a pendulum with small amplitude. The mass has horizontal velocity $v_x(t)$ and vertical velocity $v_y(t)$. Which of the following could be a graph of the curve $(v_x(t), v_y(t))$ over a complete oscillation?



Solution. (C)

Problem 2.16 ($F = ma$ 2022). The two ends of a uniform rod of length $2L$ are hung on massless strings of length L.



If the strings are attached to the ceiling, and the rod is pulled a small distance horizontally and released as shown, what is the period of oscillation?

(A) $2\pi\sqrt{\frac{L}{g}}$

(B) $2\pi\sqrt{\frac{7L}{6g}}$

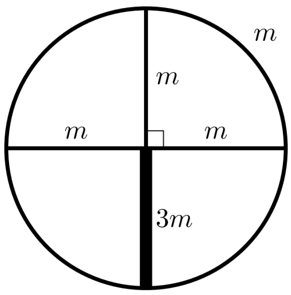
(C) $2\pi\sqrt{\frac{4L}{3g}}$

(D) $2\pi\sqrt{\frac{2L}{g}}$

(E) $2\pi\sqrt{\frac{7L}{3g}}$

Solution. (A)

Problem 2.17 ($F = ma$ 2024). A wheel of radius R has a thin rim and four spokes, each of which have uniform density.



The entire rim has mass m, three of the spokes each have mass m, and the fourth spoke has mass $3m$. The wheel is suspended on a horizontal frictionless axle passing through its center. If the wheel is slightly rotated from its equilibrium position, what is the angular frequency of small oscillations?

(A) $\sqrt{\frac{g}{3R}}$

(B) $\sqrt{\frac{g}{2R}}$

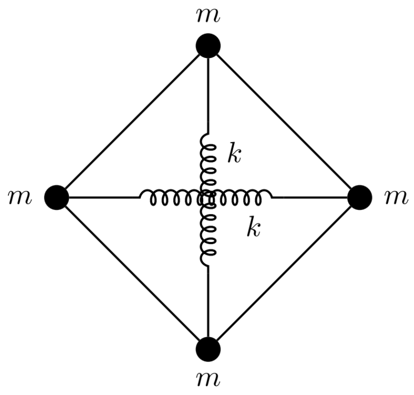
(C) $\sqrt{\frac{2g}{3R}}$

(D) $\sqrt{\frac{g}{R}}$

(E) $\sqrt{\frac{7g}{6R}}$

Solution. (A)

Problem 2.18 ($F = ma$ 2024). Four massless rigid rods are connected into a quadrilateral by four hinges. The hinges have mass m, and allow the rods to freely rotate. A spring of spring constant k is connected across each of the diagonals, so that the springs are at their relaxed length when the rods form a square.



Assume the springs do not interfere with each other. If the square is slightly compressed along one of its diagonals, its shape will oscillate over time. What is the period of these oscillations?

(A) $2\pi\sqrt{\frac{m}{4k}}$

(B) $2\pi\sqrt{\frac{m}{2k}}$

(C) $2\pi\sqrt{\frac{m}{k}}$

(D) $2\pi\sqrt{\frac{2m}{k}}$

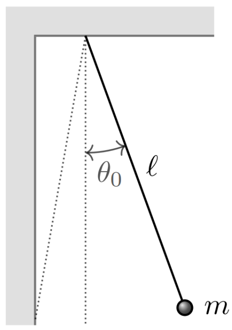
(E) $2\pi\sqrt{\frac{4m}{k}}$

Solution. (B)

Problem 2.19. A cube of side length ℓ and density ρ_c floats in water of density ρ_w with a face parallel to the water surface. The hydrostatic buoyant force by the water on the cube points upwards with a magnitude of $F = \rho_w V_{\text{disp}} g$, where V_{disp} is the volume of water displaced by the cube and g is the gravitational acceleration. The cube is given a small impulse downwards through its centre, causing it to undergo small oscillations. Find the angular frequency of oscillations, neglecting the motion of the water.

Solution. $w = \sqrt{\frac{\rho_w g}{\rho_c l}}$

Problem 2.20 ($F = ma$ 2023). A bead attached to a string of length $\ell = 10$ m is released from a very small angle θ_0 to the vertical. A wall is placed in the path of the bead such that the bead collides elastically with the wall when the string is at an angle $\frac{\theta_0}{2}$ to the vertical, as shown.



What is the time interval between the bead's collisions with the wall?

(A) $\frac{2\pi}{3}$ s

(B) $\frac{3\pi}{4}$ s

(C) $\frac{4\pi}{3}$ s

(D) $\frac{3\pi}{2}$ s

(E) 2π s

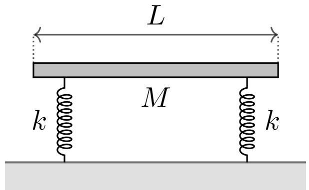
Solution. (C)

Problem 2.21 (Morin 4.20). A mass m is attached to n springs with relaxed lengths of zero. The spring constants are $k_1, k_2, \ldots, k_n$. The mass initially sits at its equilibrium position and then is given a kick in an arbitrary direction. Describe the resulting motion.

Solution. The mass will move towards its equilibrium position and execute SHM along the axis it was kicked with $w = \sqrt{\frac{\sum k_n}{m}}$

Problem 2.22 ($F = ma$ 2022). **The following information applies to the next two problems.**

A uniform rod of length L and mass M is placed with its ends resting on two identical springs of spring constant k, as shown.



The rod is initially in equilibrium. If the rod is uniformly displaced downward and released from rest, what is the frequency f of its oscillations?

(A) $\frac{1}{2\pi}\sqrt{\frac{k}{4M}}$

(B) $\frac{1}{2\pi}\sqrt{\frac{k}{2M}}$

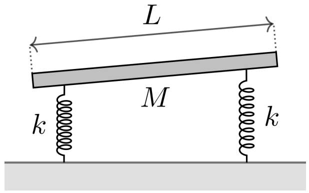
(C) $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$

(D) $\frac{1}{2\pi}\sqrt{\frac{2k}{M}}$

(E) $\frac{1}{2\pi}\sqrt{\frac{4k}{M}}$

Solution. (D)

Problem 2.23 ($F = ma$ 2022). Next, the rod is brought back to equilibrium. It is slightly rotated about its center of mass, then released from rest.



What is the frequency f of its oscillations?

(A) $\frac{1}{2\pi}\sqrt{\frac{k}{M}}$

(B) $\frac{1}{2\pi}\sqrt{\frac{2k}{M}}$

(C) $\frac{1}{2\pi}\sqrt{\frac{3k}{M}}$

(D) $\frac{1}{2\pi}\sqrt{\frac{4k}{M}}$

(E) $\frac{1}{2\pi}\sqrt{\frac{6k}{M}}$

Solution. (E)

Problem 2.24 ($F = ma$ 2025). A particle of mass m moves in the xy plane with potential energy

$$U(x, y) = kxy/2$$

If the particle begins at the origin, then it is possible to displace it slightly in some direction, so that the particle subsequently oscillates periodically. What is the period of this motion?

(A) $2\pi\sqrt{\frac{m}{4k}}$

(B) $2\pi\sqrt{\frac{m}{2k}}$

(C) $2\pi\sqrt{\frac{m}{k}}$

(D) $2\pi\sqrt{\frac{2m}{k}}$

(E) $2\pi\sqrt{\frac{4m}{k}}$

Solution. (D)

Problem 2.25 (Morin 4.22). A spring with relaxed length zero and spring constant k is attached to the ground. A projectile of mass m is attached to the other end of the spring. The projectile is then picked up and thrown with velocity v at an angle θ to the horizontal. (a) Geometrically, what kind of curve is the resulting trajectory? (b) Find the value of v so that the projectile hits the ground travelling straight downward.

Solution. a) An ellipse centred at (0,mg/k)
b) $v = \frac{g}{sin(\theta)}\sqrt{\frac{m}{k}}$